

MULTISPECIES TASEP AND THE TETRAHEDRON EQUATION

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Abstract

We introduce a family of layer to layer transfer matrices in a three-dimensional (3D) lattice model which can be viewed as partition functions of the q -oscillator valued six-vertex model on $m \times n$ square lattice. By invoking the tetrahedron equation we establish their commutativity and bilinear relations mixing various boundary conditions. At $q = 0$ and $m = n$, they ultimately yield a new proof of the steady state formula for the n -species totally asymmetric simple exclusion process (TASEP) obtained recently by the authors, revealing the 3D integrability in the matrix product construction.

1. INTRODUCTION

Totally asymmetric simple exclusion process (TASEP) is a model of non-equilibrium stochastic dynamics in physical, biological and many other systems. It has been studied extensively in the last few decades especially in one-dimension, which has led to numerous generalizations and analytical results. See for example [4, 5] and references therein.

By n -species TASEP or n -TASEP for short we mean in this paper the TASEP on a one-dimensional periodic chain \mathbb{Z}_L with L -sites in which local states σ_i take values in $\{0, 1, \dots, n\}$ and neighboring pairs $(\sigma_i, \sigma_{i+1}) = (\alpha, \beta)$ with $\alpha > \beta$ are interchanged to (β, α) with a uniform transition rate.

The main theme of the present paper, which is a continuation of [10], is the 3D integrability of the n -TASEP connected to the *tetrahedron equation* [14], a 3D generalization of the Yang-Baxter equation [2]. It becomes visible and natural for the multispecies case $n \geq 2$.

In [10], combinatorial construction of the steady state probability $\mathbb{P}(\sigma_1, \dots, \sigma_L)$ of the n -TASEP by Ferrari-Martin [8] was identified with a composition of the *combinatorial R* [12]. It is a quantum R matrix of $U_q(\widehat{sl}_L)$ at $q = 0$ where the original periodic chain \mathbb{Z}_L has been incorporated into the Dynkin diagram of the relevant quantum group. It has led to a new matrix product formula

$$\mathbb{P}(\sigma_1, \dots, \sigma_L) = \text{Tr}(X_{\sigma_1} \cdots X_{\sigma_L}) \quad (1.1)$$

by applying the recent matrix product construction of the R matrix based on the tetrahedron equation [11]. The result in [10] possesses distinct features from the other ones [7, 13, 6]. The operator X_σ itself is expressed as a configuration sum for a *corner transfer matrix* [2] of the $q = 0$ -oscillator valued five-vertex model. See (2.7). It serves as a layer to layer transfer matrix to constitute $\text{Tr}(X_{\sigma_1} \cdots X_{\sigma_L})$ as a *partition function* of a 3D lattice model.

Our aim in this paper is to elucidate a further 3D integrability concerning an alternative approach to establish (1.1). It is by the so called *hat relation*

$$\sum_{0 \leq \gamma, \delta \leq n} h_{\gamma, \delta}^{\alpha, \beta} X_\gamma X_\delta = X_\alpha \hat{X}_\beta - \hat{X}_\alpha X_\beta \quad (0 \leq \alpha, \beta \leq n), \quad (1.2)$$

where $h_{\gamma, \delta}^{\alpha, \beta}$ is an element of the local Markov matrix defined in (2.2) and (2.5). Construction of such companion operators $\hat{X}_0, \dots, \hat{X}_n$ is a sufficient task to prove (1.1)¹ as is well known [4]. See also

¹ as long as the right hand side is convergent

section 2.3. We construct \hat{X}_i similarly to X_i as a weighted configuration sum as in (2.7) and present a self-contained proof of the hat relation (1.2). Our strategy is to upgrade the statement ultimately by introducing q -deformation, spectral parameters and embedding into a 3D lattice model until the point where all the nonlocal commutation relations can be understood most naturally as a consequence of the single and local tetrahedron equation. The analysis fully demonstrates the 3D integrable aspect of the steady state in the n -TASEP as promised in [10].

The paper is organized as follows. In section 2 we recall the n -TASEP and the steady state result in [10]. The operators X_i and \hat{X}_i are defined and the main statement, the hat relation, is formulated (theorem 2.2). In section 3 we introduce the deformation parameter q and define the 3D L and M operators. Eigenvectors of the latter and the tetrahedron equation among L and M (theorem 3.4) are described. These contents serve as the local information controlling more involved nonlocal objects considered in the subsequent sections. In section 4 we consider the 3D lattice model associated with the 3D L operator. A family of layer to layer transfer matrices labeled with mixed boundary conditions $S(z)_j^a$ are introduced. It is shown that each of them form a commuting family by invoking the tetrahedron equation and the eigenvectors of M (proposition 4.5). In section 5 we extend the method in section 4 further to generate a family of bilinear relations involving the layer to layer transfer matrices with various boundary labels (theorem 5.1). They form the most general relations in this paper (see remark 5.4), which ultimately specialize to the hat relation. In section 6 we explain how the $q = 0$ case of the results in section 5 yield the difference analogue of the hat relation (proposition 6.5). The original hat relation is an immediate consequence of it as mentioned in the end. Section 7 is devoted to a summary and an outlook.

2. n -SPECIES TASEP

2.1. Definition of n -TASEP. Consider the periodic 1D chain with L sites \mathbb{Z}_L . Each site $i \in \mathbb{Z}_L$ is populated with a local state $\sigma_i \in \{0, 1, \dots, n\}$. It is interpreted as the species of the particle occupying it or 0 indicating the absence of particles. We assume $1 \leq n < L$. Consider a stochastic model on \mathbb{Z}_L such that neighboring pairs of local states $(\sigma_i, \sigma_{i+1}) = (\alpha, \beta)$ are interchanged as $\alpha\beta \rightarrow \beta\alpha$ if $\alpha > \beta$ with the uniform transition rate. The space of states is given by

$$(\mathbb{C}^{n+1})^{\otimes L} \simeq \bigoplus_{(\sigma_1, \dots, \sigma_L) \in \{0, \dots, n\}^L} \mathbb{C}|\sigma_1, \dots, \sigma_L\rangle. \quad (2.1)$$

Let $\mathbb{P}(\sigma_1, \dots, \sigma_L; t)$ be the probability of finding the configuration $(\sigma_1, \dots, \sigma_L)$ at time t , and set

$$|P(t)\rangle = \sum_{(\sigma_1, \dots, \sigma_L) \in \{0, \dots, n\}^L} \mathbb{P}(\sigma_1, \dots, \sigma_L; t) |\sigma_1, \dots, \sigma_L\rangle.$$

By n -TASEP we mean the stochastic system governed by the continuous-time master equation

$$\frac{d}{dt}|P(t)\rangle = H|P(t)\rangle,$$

where the Markov matrix has the form

$$H = \sum_{i \in \mathbb{Z}_L} h_{i, i+1}, \quad h|\alpha, \beta\rangle = \begin{cases} |\beta, \alpha\rangle - |\alpha, \beta\rangle & (\alpha > \beta), \\ 0 & (\alpha \leq \beta). \end{cases} \quad (2.2)$$

Here $h_{i, i+1}$ is the local Markov matrix that acts as h on the i -th and the $(i+1)$ -th components and as the identity elsewhere. As H preserves the particle content, it acts on each *sector* consisting of the configurations with prescribed *multiplicity* $\mathbf{m} = (m_0, \dots, m_n) \in (\mathbb{Z}_{\geq 0})^{n+1}$ of particles:

$$S(\mathbf{m}) = \{\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_L) \in \{0, \dots, n\}^L \mid \sum_{j=1}^L \delta_{k, \sigma_j} = m_k, \forall k\}.$$

The space of states (2.1) is decomposed as $\bigoplus_{\mathbf{m}} \bigoplus_{\boldsymbol{\sigma} \in S(\mathbf{m})} \mathbb{C}|\boldsymbol{\sigma}\rangle$, where the outer sum ranges over $m_i \in \mathbb{Z}_{\geq 0}$ such that $m_0 + \dots + m_n = L$. A sector $\bigoplus_{\boldsymbol{\sigma} \in S(\mathbf{m})} \mathbb{C}|\boldsymbol{\sigma}\rangle$ such that $m_i \geq 1$ for all $0 \leq i \leq n$ is called *basic*. Non-basic sectors are equivalent to a basic sector for n' -TASEP with some $n' < n$

by a suitable relabeling of species. Thus we shall exclusively deal with basic sectors in this paper, hence $n < L$ as mentioned before. This condition guarantees [10] the convergence of the right hand side of (2.4). The spectrum of H is known to exhibit a remarkable duality [1].

2.2. Steady states. In each sector $\bigoplus_{\sigma \in S(\mathbf{m})} \mathbb{C}|\sigma\rangle$ there is a unique vector $|\bar{P}(\mathbf{m})\rangle$ up to a normalization, called the *steady state*, satisfying $H|\bar{P}(\mathbf{m})\rangle = 0$. The steady state for 1-TASEP is trivial under the periodic boundary condition in that all the monomials have the same coefficient, i.e. all the configurations are realized with an equal probability.

Example 2.1. We present (unnormalized) steady states in small sectors of 2-TASEP and 3-TASEP in the form

$$|\bar{P}(\mathbf{m})\rangle = |\xi(\mathbf{m})\rangle + C|\xi(\mathbf{m})\rangle + \cdots + C^{L-1}|\xi(\mathbf{m})\rangle$$

respecting the symmetry $HC = CH$ under the \mathbb{Z}_L cyclic shift $C : |\sigma_1, \sigma_2, \dots, \sigma_L\rangle \mapsto |\sigma_L, \sigma_1, \dots, \sigma_{L-1}\rangle$. The choice of the vector $|\xi(\mathbf{m})\rangle$ is not unique.

$$\begin{aligned} |\xi(1, 1, 1)\rangle &= 2|012\rangle + |102\rangle, \\ |\xi(2, 1, 1)\rangle &= 3|0012\rangle + 2|0102\rangle + |1002\rangle, \\ |\xi(1, 2, 1)\rangle &= 2|0112\rangle + |1012\rangle + |1102\rangle, \\ |\xi(1, 1, 2)\rangle &= 3|1220\rangle + 2|2120\rangle + |2210\rangle, \\ |\xi(1, 2, 2)\rangle &= 3|11220\rangle + 2|12120\rangle + |12210\rangle + 2|21120\rangle + |21210\rangle + |22110\rangle, \\ |\xi(2, 1, 2)\rangle &= |00221\rangle + 2|02021\rangle + 3|02201\rangle + 3|20021\rangle + 5|20201\rangle + 6|22001\rangle, \\ |\xi(2, 2, 1)\rangle &= 3|00112\rangle + 2|01012\rangle + 2|01102\rangle + |10012\rangle + |10102\rangle + |11002\rangle, \\ |\xi(1, 1, 1, 1)\rangle &= 9|0123\rangle + 3|0213\rangle + 3|1023\rangle + 5|1203\rangle + 3|2013\rangle + |2103\rangle, \\ |\xi(2, 1, 1, 1)\rangle &= 24|00123\rangle + 6|00213\rangle + 12|01023\rangle + 17|01203\rangle + 8|02013\rangle + 3|02103\rangle \\ &\quad + 4|10023\rangle + 7|10203\rangle + 9|12003\rangle + 6|20013\rangle + 3|20103\rangle + |21003\rangle, \\ |\xi(1, 2, 1, 1)\rangle &= 12|01123\rangle + 5|01213\rangle + 3|02113\rangle + 4|10123\rangle + 3|10213\rangle + 4|11023\rangle \\ &\quad + 7|11203\rangle + 5|12013\rangle + 2|12103\rangle + 3|20113\rangle + |21013\rangle + |21103\rangle, \\ |\xi(1, 1, 2, 1)\rangle &= 12|01223\rangle + 5|02123\rangle + 3|02213\rangle + 3|10223\rangle + 5|12023\rangle + 7|12203\rangle \\ &\quad + 4|20123\rangle + 3|20213\rangle + |21023\rangle + 2|21203\rangle + 4|22013\rangle + |22103\rangle, \\ |\xi(1, 1, 1, 2)\rangle &= 24|12330\rangle + 12|13230\rangle + 4|13320\rangle + 6|21330\rangle + 8|23130\rangle + 6|23310\rangle \\ &\quad + 17|31230\rangle + 7|31320\rangle + 3|32130\rangle + 3|32310\rangle + 9|33120\rangle + |33210\rangle. \end{aligned}$$

As these coefficients indicate, steady states are nontrivial for $n \geq 2$. We will demonstrate the 3D integrability behind them which will ultimately be related to the tetrahedron equation.

2.3. Matrix product formula. Consider the steady state

$$|\bar{P}(\mathbf{m})\rangle = \sum_{\sigma \in S(\mathbf{m})} \mathbb{P}(\sigma) |\sigma\rangle \quad (2.3)$$

and postulate that the steady state probability $\mathbb{P}(\sigma)$ is expressed in the matrix product form

$$\mathbb{P}(\sigma_1, \dots, \sigma_L) = \text{Tr}(X_{\sigma_1} \cdots X_{\sigma_L}) \quad (2.4)$$

in terms of some operators X_0, \dots, X_n . Introduce the notations for the matrix elements of the local Markov matrix (2.2) and the associated product of X_i 's as

$$h|\alpha, \beta\rangle = \sum_{\gamma, \delta} h_{\alpha, \beta}^{\gamma, \delta} |\gamma, \delta\rangle, \quad (hXX)_{\alpha, \beta} := \sum_{\gamma, \delta} h_{\gamma, \delta}^{\alpha, \beta} X_{\gamma} X_{\delta}. \quad (2.5)$$

Then we have

$$\begin{aligned}
H|\bar{P}(\mathbf{m})\rangle &= \sum_{i \in \mathbb{Z}_L} \sum_{\sigma \in S(\mathbf{m})} \mathbb{P}(\dots, \sigma_i, \sigma_{i+1}, \dots) h_{i, i+1} |\dots, \sigma_i, \sigma_{i+1}, \dots\rangle \\
&= \sum_{i \in \mathbb{Z}_L} \sum_{\sigma \in S(\mathbf{m})} \sum_{\sigma'_i, \sigma'_{i+1}} \text{Tr}(\dots X_{\sigma_i} X_{\sigma_{i+1}} \dots) h_{\sigma_i, \sigma_{i+1}}^{\sigma'_i, \sigma'_{i+1}} |\dots, \sigma'_i, \sigma'_{i+1}, \dots\rangle \\
&= \sum_{\sigma \in S(\mathbf{m})} \sum_{i \in \mathbb{Z}_L} \text{Tr}(\dots (hXX)_{\sigma_i, \sigma_{i+1}} \dots) |\dots, \sigma_i, \sigma_{i+1}, \dots\rangle.
\end{aligned}$$

Therefore if there are another set of operators $\hat{X}_0, \dots, \hat{X}_n$ obeying the *hat relation*

$$(hXX)_{\alpha, \beta} = X_\alpha \hat{X}_\beta - \hat{X}_\alpha X_\beta, \quad (2.6)$$

the vector (2.3) satisfies $H|\bar{P}(\mathbf{m})\rangle = 0$ thanks to the cyclicity of the trace (cf. [4]). Then (2.4), if finite, must coincide with the actual steady state probability up to an overall normalization due to the uniqueness of the steady state. Note on the other hand that \hat{X}_i satisfying the hat relation with a given X_i is not unique. For instance $\hat{X}_i \rightarrow \hat{X}_i + cX_i$ leaves (2.6) unchanged.

2.4. Main result. In our previous work [10], a new matrix product formula (2.4) of the steady state probability of the n -TASEP was proved which involves the operators X_0, \dots, X_n in the left diagram of

$$X_i = \sum \text{Diagram} \quad \hat{X}_i = \sum (\alpha_1 + \dots + \alpha_n) \text{Diagram} \quad (2.7)$$

The diagrams in (2.7) represent configurations on a triangular lattice. The left diagram for X_i shows a triangular region with a boundary of length $n-i$ on the right and a boundary of length i on the bottom. The right diagram for \hat{X}_i is similar but includes weights $\alpha_1, \dots, \alpha_n$ on the top boundary.

The proof was done by identifying the Ferrari-Martin algorithm [8] with a composition of the combinatorial R . It did not rely on the hat relation, although \hat{X}_i defined by the right diagram was announced to fulfill it. The main result of this paper is a self-contained proof of the hat relation (2.6) which reads explicitly as follows:

Theorem 2.2 (Hat relation). *The operators X_i and \hat{X}_i in (2.7) satisfy*

$$\begin{aligned}
[X_i, \hat{X}_j] &= [\hat{X}_i, X_j] \quad (0 \leq i, j \leq n), \\
X_i X_j &= \hat{X}_i X_j - X_i \hat{X}_j \quad (0 \leq j < i \leq n).
\end{aligned}$$

The proof will be achieved in the end of section 6 as a consequence of its far-reaching generalization by embarking on q -deformed counterparts, layer to layer transfer matrices, their bilinear relations and so forth.

In the rest of the section we explain the definition (2.7). First we consider the X_i in the left diagram. It represents a configuration sum, i.e. the partition function of the $q = 0$ -oscillator valued five-vertex model on the triangular shape region of a square lattice with a prescribed condition along the SW-NE boundary.

$$\begin{array}{ccccc}
\begin{array}{c} 0 \\ \uparrow \\ 0 \text{---} 0 \\ \downarrow \\ 0 \end{array} & \begin{array}{c} 1 \\ \uparrow \\ 1 \text{---} 1 \\ \downarrow \\ 1 \end{array} & \begin{array}{c} 1 \\ \uparrow \\ 1 \text{---} 0 \\ \downarrow \\ 0 \end{array} & \begin{array}{c} 0 \\ \uparrow \\ 0 \text{---} 1 \\ \downarrow \\ 1 \end{array} & \begin{array}{c} 1 \\ \uparrow \\ 0 \text{---} 0 \\ \downarrow \\ 1 \end{array} \\
1 & 1 & \mathbf{a}^+ & \mathbf{a}^- & \mathbf{k}
\end{array} \quad (2.8)$$

Each edge takes 0 or 1 and the sum extends over all the configurations such that every vertex is one of the above five types. In (2.8) we have colored the edges assuming 0 and 1 in black and

red respectively. This convention will apply in the rest of the paper². Given such a configuration, the summand is the *tensor product* of the local “Boltzmann weight” $1, \mathbf{a}^+, \mathbf{a}^-, \mathbf{k}$ assigned to each vertex as specified in the above³. They are linear operators on the Fock space $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle^4$ as $(|-1\rangle = 0, 1|m\rangle = |m\rangle)$

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = |m-1\rangle, \quad \mathbf{k}|m\rangle = \delta_{m,0}|m\rangle \quad (2.9)$$

obeying the relations

$$\mathbf{k} \mathbf{a}^+ = 0, \quad \mathbf{a}^- \mathbf{k} = 0, \quad \mathbf{a}^+ \mathbf{a}^- = 1 - \mathbf{k}, \quad \mathbf{a}^- \mathbf{a}^+ = 1. \quad (2.10)$$

They are identified with the specialization of the q -oscillator algebra \mathcal{A}_q in (3.1) and (3.2) to $q = 0$. Thus we write $\mathbf{a}^+, \mathbf{a}^-, \mathbf{k} \in \mathcal{A}_{q=0}$ here. The 0-oscillator operators attached to vertices at different positions act on independent copies of the Fock space. Thus $X_i \in (\mathcal{A}_{q=0})^{\otimes n(n-1)/2} \subseteq \text{End}(F^{\otimes n(n-1)/2})$. Accordingly the trace in (2.4) is taken over $F^{\otimes n(n-1)/2}$. In each component it is calculated by $\text{Tr}_F(X) = \sum_{m \geq 0} \langle m|X|m\rangle$ with $\langle m|m'\rangle = \delta_{m,m'}$.

Remark 2.3. Our result (2.4) with (2.7) corresponds to the *integer normalization*

$$\mathbb{P}(\sigma_1, \dots, \sigma_L) = 1 \quad \text{for } \sigma_1 \geq \dots \geq \sigma_L.$$

In this normalization $\mathbb{P}(\sigma) \in \mathbb{Z}_{\geq 1}$ holds for all the state $\sigma \in S(\mathbf{m})$. These facts can be shown via the equivalent formula [10, eq.(4.3)] and the NY-rule for the combinatorial R explained in [10, sec.2.4]. Example 2.1 has been given in this normalization.

The X_i has the form of a *corner transfer matrix* [2] of the 0-oscillator valued five-vertex model, although it acts along the perpendicular direction to the layer as opposed to the usual 2D setting. Equivalently one may view it as a layer to layer transfer matrix of the 3D lattice model where F is assigned with the edges perpendicular to the plane on which the five-vertex model is defined. The steady state probability (2.4) is then interpreted as a *partition function* of the 3D system of prism shape which is periodic along the third direction.

As for the \hat{X}_i in the right diagram of (2.7), it means a similar configuration sum but now weighted by the coefficient $\alpha_1 + \dots + \alpha_n$.

Example 2.4. For $n = 2$ the operator X_i is given by

$$X_0 = \begin{array}{c} \uparrow \uparrow \\ \text{---} \end{array} + \begin{array}{c} \uparrow \uparrow \\ \text{---} \end{array} = 1 + \mathbf{a}^+, \quad X_1 = \begin{array}{c} \uparrow \uparrow \\ \text{---} \end{array} = \mathbf{k}, \quad X_2 = \begin{array}{c} \uparrow \uparrow \\ \text{---} \end{array} + \begin{array}{c} \uparrow \uparrow \\ \text{---} \end{array} = \mathbf{a}^- + 1.$$

Accordingly we have $\hat{X}_0 = \mathbf{a}^+, \hat{X}_1 = \mathbf{k}, \hat{X}_2 = \mathbf{a}^- + 2$. Thus for instance,

$$\mathbb{P}(20201) = \text{Tr}(X_2 X_0 X_2 X_0 X_1) = \text{Tr}((1 + \mathbf{a}^+)(1 + \mathbf{a}^-)(1 + \mathbf{a}^+)(1 + \mathbf{a}^-)\mathbf{k}) = 5$$

reproducing the second last term in $|\xi(2, 1, 2)\rangle$ in example 2.1.

Example 2.5. For $n = 3$ the operator X_i is given by

$$\begin{aligned} X_0 = & \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{---} \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{---} \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{---} \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{---} \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ \text{---} \end{array} \\ & = 1 \otimes 1 \otimes 1 + \mathbf{a}^+ \otimes 1 \otimes 1 + \mathbf{k} \otimes \mathbf{a}^+ \otimes 1 + \mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{a}^+ + 1 \otimes \mathbf{a}^+ \otimes \mathbf{a}^+, \end{aligned}$$

²Although, in the formula like (2.7), the black edges not on the SW-NE boundary should be understood as taking both 0 and 1.

³At the boundary corners in (2.7) where arrows make 90° left turns, we assume no change in the edge states and assign the weight 1. See examples 2.4 and 2.5.

⁴The ket vector here should not be confused with the TASEP states in section 2.1-2.4.

$$\begin{aligned}
X_1 &= \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \end{array} \\
&= \mathbf{k} \otimes \mathbf{k} \otimes 1 + \mathbf{a}^- \otimes \mathbf{k} \otimes \mathbf{a}^+ + 1 \otimes \mathbf{k} \otimes \mathbf{a}^+, \\
X_2 &= \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \end{array} \\
&= 1 \otimes \mathbf{a}^- \otimes \mathbf{k} + \mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{k} + \mathbf{k} \otimes 1 \otimes \mathbf{k}, \\
X_3 &= \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \downarrow \quad \downarrow \quad \downarrow \end{array} \\
&= 1 \otimes \mathbf{a}^- \otimes \mathbf{a}^- + \mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{a}^- + \mathbf{k} \otimes 1 \otimes \mathbf{a}^- + \mathbf{a}^- \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1.
\end{aligned}$$

Here and in what follows, the components of the tensor product will always be ordered so that they correspond, from left to right, to the vertices (if exist) at $(1, 1)$, $(2, 1)$, $(1, 2)$, $(3, 1)$, $(2, 2)$, $(1, 3)$, \dots , where (i, j) is the intersection of the i -th horizontal line from the top and the j -th vertical line from the left. Accordingly we have

$$\begin{aligned}
\hat{X}_0 &= \mathbf{a}^+ \otimes 1 \otimes 1 + \mathbf{k} \otimes \mathbf{a}^+ \otimes 1 + \mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{a}^+ + 2(1 \otimes \mathbf{a}^+ \otimes \mathbf{a}^+), \\
\hat{X}_1 &= \mathbf{k} \otimes \mathbf{k} \otimes 1 + \mathbf{a}^- \otimes \mathbf{k} \otimes \mathbf{a}^+ + 2(1 \otimes \mathbf{k} \otimes \mathbf{a}^+), \\
\hat{X}_2 &= 1 \otimes \mathbf{a}^- \otimes \mathbf{k} + 2\mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{k} + 2\mathbf{k} \otimes 1 \otimes \mathbf{k}, \\
\hat{X}_3 &= 1 \otimes \mathbf{a}^- \otimes \mathbf{a}^- + 2\mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{a}^- + 2\mathbf{k} \otimes 1 \otimes \mathbf{a}^- + 2\mathbf{a}^- \otimes 1 \otimes 1 + 3(1 \otimes 1 \otimes 1).
\end{aligned}$$

3. 3D L, M OPERATORS AND THE TETRAHEDRON EQUATION

In this section, we define L and M operators involving a generic parameter q and describe their properties used in later sections.

3.1. q -oscillator algebra and the Fock space. Let q be a generic complex parameter unless it is set to be 0 in section 6. Let \mathcal{A}_q be the q -oscillator algebra generated by $\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}$ with relations

$$\mathbf{k} \mathbf{a}^\pm = -q^{\pm 1} \mathbf{a}^\pm \mathbf{k}, \quad \mathbf{a}^+ \mathbf{a}^- = 1 - \mathbf{k}^2, \quad \mathbf{a}^- \mathbf{a}^+ = 1 - q^2 \mathbf{k}^2. \quad (3.1)$$

We also consider $\tilde{\mathcal{A}}_q = \mathcal{A}_{-q}$ generated by $\mathbf{a}^+, \mathbf{a}^-, \tilde{\mathbf{k}}$ with relations

$$\tilde{\mathbf{k}} \mathbf{a}^\pm = q^{\pm 1} \mathbf{a}^\pm \tilde{\mathbf{k}}, \quad \mathbf{a}^+ \mathbf{a}^- = 1 - \tilde{\mathbf{k}}^2, \quad \mathbf{a}^- \mathbf{a}^+ = 1 - q^2 \tilde{\mathbf{k}}^2.$$

\mathcal{A}_q and $\tilde{\mathcal{A}}_q$ act on the Fock space $F = \bigoplus_{m \geq 0} \mathbb{C}|m\rangle$ as⁵

$$\mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1 - q^{2m})|m-1\rangle, \quad \mathbf{k}|m\rangle = (-q)^m|m\rangle, \quad \tilde{\mathbf{k}}|m\rangle = q^m|m\rangle. \quad (3.2)$$

We define the dual Fock space $F^* = \bigoplus_{m \geq 0} \mathbb{C}\langle m|$ on which \mathcal{A}_q and $\tilde{\mathcal{A}}_q$ act from right as

$$\langle m|\mathbf{a}^+ = (1 - q^{2m})\langle m-1|, \quad \langle m|\mathbf{a}^- = \langle m+1|, \quad \langle m|\mathbf{k} = (-q)^m\langle m|, \quad \langle m|\tilde{\mathbf{k}} = q^m\langle m|.$$

The pairing $F^* \otimes F \rightarrow \mathbb{C}$ is determined as $\langle m|m'\rangle = (q^2)_m \delta_{m,m'}$ with $(q)_m = \prod_{1 \leq j \leq m} (1 - q^j)$ so as to satisfy $(\langle m|X)|m'\rangle = \langle m|(X|m'\rangle)$ for any $X \in \mathcal{A}_q, \tilde{\mathcal{A}}_q$.

We finally prepare the two-dimensional vector space V and its dual V^* by

$$V = \mathbb{C}v_0 \oplus \mathbb{C}v_1, \quad V^* = \mathbb{C}v_0^* \oplus \mathbb{C}v_1^*, \quad \langle v_i^*, v_j \rangle = \delta_{ij}. \quad (3.3)$$

⁵We warn that the same notation $\mathbf{a}^\pm, \mathbf{k}$ and F will be used either for $q = 0$ or not.

3.2. 3D L, M operators with spectral parameter. We introduce 3D L, M operators [3] with spectral parameter z . They are linear operators on $V \otimes V \otimes F$. For $i, j \in \{0, 1\}$ and $|\xi\rangle \in F$, define $\mathcal{L}(z)$ by

$$\mathcal{L}(z)(v_i \otimes v_j \otimes |\xi\rangle) = \sum_{a,b=0,1} v_a \otimes v_b \otimes \mathcal{L}(z)_{i,j}^{a,b} |\xi\rangle, \quad (3.4)$$

where $\mathcal{L}_{i,j}^{a,b}(z)$ is an operator on F such that

$$\mathcal{L}(z)_{0,0}^{0,0} = \mathcal{L}(z)_{1,1}^{1,1} = 1, \quad \mathcal{L}(z)_{1,0}^{0,1} = z\mathbf{a}^+, \quad \mathcal{L}(z)_{0,1}^{1,0} = z^{-1}\mathbf{a}^-, \quad \mathcal{L}(z)_{0,1}^{0,1} = \mathbf{k}, \quad \mathcal{L}(z)_{1,0}^{1,0} = q\mathbf{k}. \quad (3.5)$$

The other $\mathcal{L}_{i,j}^{a,b}(z)$'s are set to be 0. One can let $\mathcal{L}(z)$ act from right on $V^* \otimes V^* \otimes F^*$ as

$$(v_a^* \otimes v_b^* \otimes \langle \xi |) \mathcal{L}(z) = \sum_{i,j=0,1} v_i^* \otimes v_j^* \otimes \langle \xi | \mathcal{L}(z)_{i,j}^{a,b}.$$

$\mathcal{M}(z)$ is defined similarly with

$$\mathcal{M}(z)_{0,0}^{0,0} = \mathcal{M}(z)_{1,1}^{1,1} = 1, \quad \mathcal{M}(z)_{1,0}^{0,1} = z\mathbf{a}^+, \quad \mathcal{M}(z)_{0,1}^{1,0} = z^{-1}\mathbf{a}^-, \quad \mathcal{M}(z)_{0,1}^{0,1} = \tilde{\mathbf{k}}, \quad \mathcal{M}(z)_{1,0}^{1,0} = -q\tilde{\mathbf{k}}.$$

Remark that the L operator in [10, eq.(2.9)] corresponds to the $z = 1$ case of the present L operator equipped with the spectral parameter z . Graphically they are expressed as follows:

$\mathcal{L}(z)_{i,j}^{a,b}$	1	1	$z\mathbf{a}^+$	$z^{-1}\mathbf{a}^-$	\mathbf{k}	$q\mathbf{k}$
$\mathcal{M}(z)_{i,j}^{a,b}$	1	1	$z\mathbf{a}^+$	$z^{-1}\mathbf{a}^-$	$\tilde{\mathbf{k}}$	$-q\tilde{\mathbf{k}}$

(3.6)

Note that z is not exhibited in the diagrams for simplicity. In view of the property

$$\mathcal{L}(z)_{i,j}^{a,b} = \mathcal{M}(z)_{i,j}^{a,b} = 0 \text{ unless } a + b = i + j, \quad (3.7)$$

$\mathcal{L}(z)$ and $\mathcal{M}(z)$ can be considered to define q -oscillator valued six-vertex models on the 2D lattice. Alternatively, we can regard $\mathcal{L}(z)$ and $\mathcal{M}(z)$ as vertices on the 3D square lattice as

$\mathcal{L}(z)_{i,j}^{a,b}$	$\mathcal{M}(z)_{i,j}^{a,b}$

(3.8)

Here, along the blue or green line runs the Fock space F . We use the two colors to distinguish $\mathcal{L}(z)$ from $\mathcal{M}(z)$.

3.3. Right and left eigenvectors of the M operator. Let us provide some right and left eigenvectors of $\mathcal{M}(z)$ for later use.

Proposition 3.1. *Set $|\chi(z)\rangle = \sum_{m \geq 0} \frac{z^m}{(q)_m} |m\rangle$. Then the following vectors are right eigenvectors of $\mathcal{M}(z)$ with eigenvalue 1 for any $|\xi\rangle \in F^*$ and $\alpha, \beta \in \mathbb{C}$.*

$$v_0 \otimes v_0 \otimes |\xi\rangle, \quad v_1 \otimes v_1 \otimes |\xi\rangle, \quad (\alpha v_1 \otimes v_0 + \beta v_0 \otimes v_1) \otimes |\chi(\frac{\alpha z}{\beta})\rangle.$$

Proof. The first two are obvious. The last one is verified by directly checking

$$\sum_{i+j=1} \alpha^i \beta^j \mathcal{M}(z)_{i,j}^{k,l} |\chi(\frac{\alpha z}{\beta})\rangle = \alpha^k \beta^l |\chi(\frac{\alpha z}{\beta})\rangle. \quad (3.9)$$

□

Similarly, we have

Proposition 3.2. *Set $\langle \chi(z) | = \sum_{m \geq 0} \frac{z^m}{(q)_m} \langle m |$. Then the following vectors are left eigenvectors of $\mathcal{M}(z)$ with eigenvalue 1 for any $\langle \xi | \in F^*$ and $\alpha, \beta \in \mathbb{C}$.*

$$v_0^* \otimes v_0^* \otimes \langle \xi |, \quad v_1^* \otimes v_1^* \otimes \langle \xi |, \quad (\alpha v_1^* \otimes v_0^* + \beta v_0^* \otimes v_1^*) \otimes \langle \chi(\frac{\alpha}{\beta z}) |.$$

Proof. Again the first two are trivial and the last one is due to

$$\sum_{i+j=1} \alpha^i \beta^j \langle \chi(\frac{\alpha}{\beta z}) | \mathcal{M}(z)_{k,l}^{i,j} = \alpha^k \beta^l \langle \chi(\frac{\alpha}{\beta z}) |. \quad (3.10)$$

□

The above propositions imply

Corollary 3.3. $(v_0 + v_1)^{\otimes 2} \otimes |\chi(z)\rangle$ (resp. $(v_0^* + v_1^*)^{\otimes 2} \otimes \langle \chi(z^{-1}) |$) is also a right (resp. left) eigenvector of $\mathcal{M}(z)$ of eigenvalue 1, i.e.,

$$\sum_{i,j} \mathcal{M}(z)_{i,j}^{k,l} |\chi(z)\rangle = |\chi(z)\rangle, \quad \langle \chi(z^{-1}) | \sum_{i,j} \mathcal{M}(z)_{k,l}^{i,j} = \langle \chi(z^{-1}) | \quad (3.11)$$

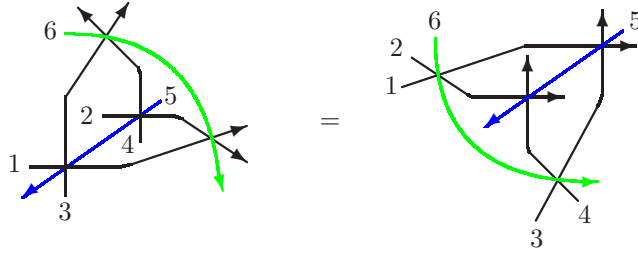
hold for any $k, l = 0, 1$.

3.4. Tetrahedron equation. The L, M operators introduced in section 3.2 satisfy the tetrahedron equation.

Theorem 3.4 (Tetrahedron equation). *As an operator on $V^{\otimes 4} \otimes F^{\otimes 2}$ the following equality holds:*

$$\mathcal{M}_{126}(z_{12}) \mathcal{M}_{346}(z_{34}) \mathcal{L}_{135}(z_{13}) \mathcal{L}_{245}(z_{24}) = \mathcal{L}_{245}(z_{24}) \mathcal{L}_{135}(z_{13}) \mathcal{M}_{346}(z_{34}) \mathcal{M}_{126}(z_{12}), \quad (3.12)$$

where $z_{ij} = z_i/z_j$. Graphically it looks as



Proof. For instance, we have

$$\begin{aligned} & \langle v_0^* \otimes v_1^* \otimes v_0^* \otimes v_1^*, (\text{LHS}) v_1 \otimes v_1 \otimes v_0 \otimes v_0 \rangle \\ &= \mathcal{L}(z_{13})_{10}^{10} \mathcal{L}(z_{24})_{10}^{01} \otimes \mathcal{M}(z_{12})_{10}^{01} \mathcal{M}(z_{34})_{01}^{01} + \mathcal{L}(z_{13})_{10}^{01} \mathcal{L}(z_{24})_{10}^{10} \otimes \mathcal{M}(z_{12})_{01}^{01} \mathcal{M}(z_{34})_{10}^{01} \\ &= (q\mathbf{k} \cdot z_{24}\mathbf{a}^+) \otimes (z_{12}\mathbf{a}^+ \cdot \tilde{\mathbf{k}}) + (z_{13}\mathbf{a}^+ \cdot q\mathbf{k}) \otimes (\tilde{\mathbf{k}} \cdot z_{34}\mathbf{a}^+) = 0 \end{aligned}$$

on $F^{\otimes 2}$, where the pairing is evaluated between $(V^*)^{\otimes 4}$ and $V^{\otimes 4}$. On the other hand, one clearly has $\langle v_0^* \otimes v_1^* \otimes v_0^* \otimes v_1^*, (\text{RHS}) v_1 \otimes v_1 \otimes v_0 \otimes v_0 \rangle = 0$. The other cases can be shown similarly. □

The above type of the tetrahedron equation was first considered in [3]. In fact, our solutions $\mathcal{L}(z), \mathcal{M}(z)$ of the tetrahedron equation are equivalent to that in [3] with a certain specialization of their spectral parameters, up to a gauge transformation of the form

$$\mathcal{L}(z) \longrightarrow P_1(\alpha) P_2(\beta) \mathcal{L}(z) P_1(\alpha')^{-1} P_2(\beta')^{-1}$$

and similarly for $\mathcal{M}(z)$, where $P_i(\gamma)$ acts nontrivially only on the i -th V . We will see that the tetrahedron equation (3.12) plays the most fundamental role controlling the whole family of relations among layer to layer transfer matrices and ultimately the hat relation in theorem 2.2.

4. LAYER TO LAYER TRANSFER MATRIX

Here we study the partition functions of the q -oscillator valued six-vertex model with special boundary conditions. Put in another way, they are layer to layer transfer matrices of a 3D lattice model whose basic unit is the 3D L operator.

Fixing positive integers m, n , we define a linear operator $T(z)$ on $V^{\otimes m} \otimes V^{\otimes n} \otimes F^{\otimes mn}$ graphically as follows:

$$T(z) = \begin{array}{c} \overbrace{\begin{array}{c} \uparrow \uparrow \dots \uparrow \uparrow \\ \rightarrow \rightarrow \rightarrow \rightarrow \\ \vdots \\ \rightarrow \rightarrow \rightarrow \rightarrow \end{array}}^n \left. \begin{array}{c} \rightarrow \rightarrow \rightarrow \rightarrow \\ \rightarrow \rightarrow \rightarrow \rightarrow \\ \vdots \\ \rightarrow \rightarrow \rightarrow \rightarrow \end{array} \right\} m \end{array}$$

Each line, horizontal or vertical, carries V (3.3). Each vertex represents $\mathcal{L}(z)_{i,j}^{a,b}$ in (3.6) including the spectral parameter z . Penetrating each vertex from back to face, the Fock space F runs along a blue line as in the left figure in (3.8). When this feature is to be emphasized, we depict $T(z)$, say for $(m, n) = (3, 4)$, as

$$T(z) = \begin{array}{c} \begin{array}{c} \uparrow \uparrow \uparrow \uparrow \\ \rightarrow \rightarrow \rightarrow \rightarrow \\ \rightarrow \rightarrow \rightarrow \rightarrow \\ \rightarrow \rightarrow \rightarrow \rightarrow \end{array} \end{array}$$

Introduce the following notation:

$$\begin{aligned} |\mathbf{i}\rangle &= v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_m}, & |\mathbf{j}\rangle &= v_{j_1} \otimes v_{j_2} \otimes \dots \otimes v_{j_n}, \\ \langle \mathbf{a}| &= v_{a_1}^* \otimes v_{a_2}^* \otimes \dots \otimes v_{a_m}^*, & \langle \mathbf{b}| &= v_{b_1}^* \otimes v_{b_2}^* \otimes \dots \otimes v_{b_n}^*, \end{aligned}$$

where all subscripts i_1, i_2, \dots , etc are 0 or 1. Then $T(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} = (\langle \mathbf{a}| \otimes \langle \mathbf{b}|) T(z) (|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) \in \text{End}(F^{\otimes mn})$ is represented as

$$T(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} = \begin{array}{c} \begin{array}{c} b_1 \ b_2 \ \dots \ b_n \\ \uparrow \uparrow \dots \uparrow \uparrow \\ i_1 \rightarrow a_1 \\ i_2 \rightarrow a_2 \\ \vdots \\ i_m \rightarrow a_m \\ j_1 \ j_2 \ \dots \ j_n \end{array} \end{array}$$

where the sums are taken over $\{0, 1\}$ for all the internal edges. With this notation, fixing $\langle \mathbf{a}|, |\mathbf{j}\rangle$ we set

$$S(z)_{\mathbf{j}}^{\mathbf{a}} = \sum_{\mathbf{i}, \mathbf{b}} T(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} = \sum_{\mathbf{i}, \mathbf{b}} \begin{array}{c} \begin{array}{c} b_1 \ b_2 \ \dots \ b_n \\ \uparrow \uparrow \dots \uparrow \uparrow \\ i_1 \rightarrow a_1 \\ i_2 \rightarrow a_2 \\ \vdots \\ i_m \rightarrow a_m \\ j_1 \ j_2 \ \dots \ j_n \end{array} \end{array} \in \text{End}(F^{\otimes mn}). \quad (4.1)$$

The operators $T(z)$, $T(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}$ and $S(z)_{\mathbf{j}}^{\mathbf{a}}$ are the layer to layer transfer matrices of size $m \times n$ with free, fixed and mixed (NW-free and SE-fixed) boundary conditions, respectively.

Example 4.1. Consider the case $(m, n) = (1, 1)$. Then we have $T(z)_{i,j}^{a,b} = \mathcal{L}(z)_{i,j}^{a,b}$, therefore

$$S(z)_0^0 = 1 + z\mathbf{a}^+, \quad S(z)_1^1 = 1 + z^{-1}\mathbf{a}^-, \quad S(z)_1^0 = \mathbf{k}, \quad S(z)_0^1 = q\mathbf{k}.$$

Example 4.2. Consider the case $(m, n) = (1, 2)$. We list those $S(z)_j^a$ which will be used in example 5.3.

$$\begin{aligned}
S_{00}^0(z) &= \begin{array}{c} \uparrow \uparrow \\ | | \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} = 1 \otimes 1 + z \mathbf{a}^+ \otimes 1 + z q \mathbf{k} \otimes \mathbf{a}^+, \\
S_{10}^0(z) &= \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} = \mathbf{k} \otimes 1 + \mathbf{a}^- \otimes \mathbf{a}^+ + z 1 \otimes \mathbf{a}^+, \\
S_{10}^1(z) &= \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} = q z^{-1} \mathbf{a}^- \otimes \mathbf{k} + q 1 \otimes \mathbf{k}, \\
S_{00}^1(z) &= \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} = q^2 \mathbf{k} \otimes \mathbf{k}.
\end{aligned}$$

Example 4.3. Consider the case $(m, n) = (2, 2)$. $S(z)_{00}^{00}$ consists of the following 8 terms:

$$\begin{array}{c} \uparrow \uparrow \\ | | \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array}$$

Thus we have

$$\begin{aligned}
S(z)_{00}^{00} &= 1 \otimes 1 \otimes 1 \otimes 1 + z \mathbf{a}^+ \otimes 1 \otimes 1 \otimes 1 + z \mathbf{k} \otimes \mathbf{a}^+ \otimes 1 \otimes 1 + z \mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{a}^+ \otimes 1 \\
&\quad + z^2 1 \otimes \mathbf{a}^+ \otimes \mathbf{a}^+ \otimes 1 + q z 1 \otimes \mathbf{k} \otimes \mathbf{k} \otimes \mathbf{a}^+ + q z^2 \mathbf{a}^+ \otimes \mathbf{k} \otimes \mathbf{k} \otimes \mathbf{a}^+ + q z \mathbf{k} \otimes 1 \otimes \mathbf{a}^+ \otimes 1.
\end{aligned}$$

Example 4.4. Similarly $S(z)_{10}^{10}$ for $(m, n) = (2, 2)$ consists of the following 8 terms:

$$\begin{array}{c} \uparrow \uparrow \\ | | \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array} + \begin{array}{c} \uparrow \uparrow \\ \color{red}{|} \color{red}{|} \\ \hline \rightarrow \end{array}$$

Thus we have

$$\begin{aligned}
S(z)_{10}^{10} &= z^{-1} 1 \otimes \mathbf{a}^- \otimes \mathbf{a}^- \otimes \mathbf{a}^+ + \mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{a}^- \otimes \mathbf{a}^+ + \mathbf{k} \otimes 1 \otimes \mathbf{a}^- \otimes \mathbf{a}^+ + \mathbf{a}^- \otimes 1 \otimes 1 \otimes \mathbf{a}^+ \\
&\quad + z 1 \otimes 1 \otimes 1 \otimes \mathbf{a}^+ + q 1 \otimes \mathbf{k} \otimes \mathbf{k} \otimes 1 + q \mathbf{k} \otimes \mathbf{a}^- \otimes 1 \otimes \mathbf{a}^+ + q z^{-1} \mathbf{a}^- \otimes \mathbf{k} \otimes \mathbf{k} \otimes 1.
\end{aligned}$$

The layer to layer transfer matrices $S(z)_j^a$ with the common SE boundary condition \mathbf{a}, \mathbf{j} form a commuting family.

Proposition 4.5 (Commutativity of layer to layer transfer matrices).

$$[S(x)_j^a, S(y)_j^a] = 0. \quad (4.2)$$

Proof. This is a consequence of the tetrahedron equation in theorem 3.4 and the ‘trivial’ eigenvectors of $\mathcal{M}(z)$ in propositions 3.1 and 3.2. Consider the following two operators on $F^{\otimes mn} \otimes F$.

$$\sum_{\mathbf{b}, \mathbf{b}'} \left(\mathcal{M}\left(\frac{x}{x'}\right)_{a_m, a_m}^{a_m, a_m} \cdots \mathcal{M}\left(\frac{x}{x'}\right)_{a_1, a_1}^{a_1, a_1} \right) \left(\mathcal{M}\left(\frac{y}{y'}\right)_{b_n, b_n}^{c_n, c_n'} \cdots \mathcal{M}\left(\frac{y}{y'}\right)_{b_1, b_1}^{c_1, c_1'} \right) T\left(\frac{x}{y}\right)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} T\left(\frac{x'}{y'}\right)_{\mathbf{i}', \mathbf{j}'}^{\mathbf{a}, \mathbf{b}'}, \quad (4.3)$$

$$\sum_{\mathbf{k}, \mathbf{k}'} T\left(\frac{x'}{y'}\right)_{\mathbf{k}', \mathbf{j}'}^{\mathbf{a}, \mathbf{c}'} T\left(\frac{x}{y}\right)_{\mathbf{k}, \mathbf{j}}^{\mathbf{a}, \mathbf{c}} \left(\mathcal{M}\left(\frac{y}{y'}\right)_{j_n, j_n}^{j_n, j_n} \cdots \mathcal{M}\left(\frac{y}{y'}\right)_{j_1, j_1}^{j_1, j_1} \right) \left(\mathcal{M}\left(\frac{x}{x'}\right)_{i_m, i_m}^{k_m, k_m'} \cdots \mathcal{M}\left(\frac{x}{x'}\right)_{i_1, i_1}^{k_1, k_1'} \right), \quad (4.4)$$

where $\mathbf{i} = (i_1, \dots, i_m)$, etc. The left block of \mathcal{M} both in (4.3) and (4.4) are actually the identities but it is better to keep them temporarily for the proof. The operators in (4.3) and (4.4) coincide.

To see this we depict them as follows.

$$\sum_{\mathbf{b}, \mathbf{b}'} \left(\text{Diagram 1} \right) = \sum_{\mathbf{k}, \mathbf{k}'} \left(\text{Diagram 2} \right) \quad (4.5)$$

Here $T(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}$ acts on $F^{\otimes mn}$ (blue arrows) and $\mathcal{M}(z)_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}$ acts on the extra single Fock space F (green arrow). In the left figure, the front and the back layers correspond to $T(\frac{x}{y})_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}}$ and $T(\frac{x'}{y'})_{\mathbf{i}', \mathbf{j}'}^{\mathbf{a}, \mathbf{b}'}$ in (4.3), respectively. Similarly in the right figure, the front and the back layers represent $T(\frac{x'}{y'})_{\mathbf{k}', \mathbf{j}'}^{\mathbf{a}, \mathbf{c}'}$ and $T(\frac{x}{y})_{\mathbf{k}, \mathbf{j}}^{\mathbf{a}, \mathbf{c}}$ in (4.4), respectively. From the top right corner of the left figure, using theorem 3.4 one can move the green line all the way down to the bottom left. It means that the left figure and the right one are equal as operators on $F^{\otimes mn} \otimes F$. Now we rephrase (4.3)=(4.4) as

$$\begin{aligned} & \sum_{\mathbf{b}, \mathbf{b}'} \left(\mathcal{M}(\frac{y}{y'})_{b_n, b'_n}^{c_n, c'_n} \cdots \mathcal{M}(\frac{y}{y'})_{b_1, b'_1}^{c_1, c'_1} \right) T(\frac{x}{y})_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} T(\frac{x'}{y'})_{\mathbf{i}', \mathbf{j}'}^{\mathbf{a}, \mathbf{b}'} \\ &= \sum_{\mathbf{k}, \mathbf{k}'} T(\frac{x'}{y'})_{\mathbf{k}', \mathbf{j}'}^{\mathbf{a}, \mathbf{c}'} T(\frac{x}{y})_{\mathbf{k}, \mathbf{j}}^{\mathbf{a}, \mathbf{c}} \left(\mathcal{M}(\frac{x}{x'})_{i_m, i'_m}^{k_m, k'_m} \cdots \mathcal{M}(\frac{x}{x'})_{i_1, i'_1}^{k_1, k'_1} \right) \end{aligned} \quad (4.6)$$

removing the identity parts. Evaluate (4.6) between $\langle \chi(\frac{y}{y'}) | \in F^*$ and $|\chi(\frac{x}{x'}) \rangle \in F$, where these vectors are on the green arrows on which only the block of $\mathcal{M}(z)$'s act. Further taking the sum over $\mathbf{i}, \mathbf{i}', \mathbf{c}, \mathbf{c}'$ on the both sides by means of (3.11) we find

$$\langle \chi(\frac{y}{y'}) | \chi(\frac{x}{x'}) \rangle \sum_{\mathbf{i}, \mathbf{i}', \mathbf{b}, \mathbf{b}'} T(\frac{x}{y})_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}, \mathbf{b}} T(\frac{x'}{y'})_{\mathbf{i}', \mathbf{j}'}^{\mathbf{a}, \mathbf{b}'} = \langle \chi(\frac{y}{y'}) | \chi(\frac{x}{x'}) \rangle \sum_{\mathbf{k}, \mathbf{k}', \mathbf{c}, \mathbf{c}'} T(\frac{x'}{y'})_{\mathbf{k}', \mathbf{j}'}^{\mathbf{a}, \mathbf{c}'} T(\frac{x}{y})_{\mathbf{k}, \mathbf{j}}^{\mathbf{a}, \mathbf{c}}.$$

Since $\langle \chi(\frac{y}{y'}) | \chi(\frac{x}{x'}) \rangle = \sum_{m \geq 0} \frac{(q^2)_m}{(q)_m^2} (\frac{xy}{x'y'})^m \neq 0$, we get $S(\frac{x}{y})_{\mathbf{j}}^{\mathbf{a}} S(\frac{x'}{y'})_{\mathbf{j}'}^{\mathbf{a}} = S(\frac{x'}{y'})_{\mathbf{j}'}^{\mathbf{a}} S(\frac{x}{y})_{\mathbf{j}}^{\mathbf{a}}$ by (4.1). \square

Example 4.6. The commutativity (4.2) is easily seen for those $S(z)_{\mathbf{j}}^{\mathbf{a}}$ in examples 4.1 and 4.2. Let us check it for $S_{00}^{00}(z) = \sum_{i=0}^2 z^i W_i$ in example 4.3. The relation $[W_i, W_j] = 0$ to be shown is nontrivial only for $(i, j) = (1, 2)$. We have $W_2 = 1 \otimes \mathbf{a}^+ \otimes \mathbf{a}^+ \otimes 1 + q\mathbf{a}^+ \otimes \mathbf{k} \otimes \mathbf{k} \otimes \mathbf{a}^+$ and $W_1 = \mathbf{a}^+ \otimes 1 \otimes 1 \otimes 1 + U \otimes 1 + qV \otimes \mathbf{a}^+$, where $U = \mathbf{k} \otimes \mathbf{a}^+ \otimes 1 + \mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{a}^+ + q\mathbf{k} \otimes 1 \otimes \mathbf{a}^+$ and $V = 1 \otimes \mathbf{k} \otimes \mathbf{k}$. As $[\mathbf{a}^+ \otimes 1 \otimes 1 \otimes 1, W_2] = 0$, we are to show

$$\begin{aligned} 0 &= [U \otimes 1 + qV \otimes \mathbf{a}^+, 1 \otimes \mathbf{a}^+ \otimes \mathbf{a}^+ \otimes 1 + q\mathbf{a}^+ \otimes \mathbf{k} \otimes \mathbf{k} \otimes \mathbf{a}^+] \\ &= [U, 1 \otimes \mathbf{a}^+ \otimes \mathbf{a}^+] \otimes 1 + qY \otimes \mathbf{a}^+ + q^2[V, \mathbf{a}^+ \otimes \mathbf{k} \otimes \mathbf{k}] \otimes (\mathbf{a}^+)^2, \end{aligned}$$

where $Y = [U, \mathbf{a}^+ \otimes \mathbf{k} \otimes \mathbf{k}] + [V, 1 \otimes \mathbf{a}^+ \otimes \mathbf{a}^+]$. Obviously the leftmost and the rightmost commutators in the last expression vanish. Hence we are to show $Y = 0$. The relation $\mathbf{k}\mathbf{a}^+ = -q\mathbf{a}^+\mathbf{k}$ in (3.1) tells $[\mathbf{k} \otimes \mathbf{a}^+ \otimes 1, \mathbf{a}^+ \otimes \mathbf{k} \otimes \mathbf{k}] = [\mathbf{k} \otimes 1 \otimes \mathbf{a}^+, \mathbf{a}^+ \otimes \mathbf{k} \otimes \mathbf{k}] = 0$. Thus $Y = 0$ reduces to $[\mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{a}^+, \mathbf{a}^+ \otimes \mathbf{k} \otimes \mathbf{k}] + [V, 1 \otimes \mathbf{a}^+ \otimes \mathbf{a}^+] = 0$. This is straightforward by (3.1).

5. FURTHER BILINEAR RELATIONS

In the proof of proposition 4.5 we used the fact that $v_i \otimes v_i \otimes |\xi\rangle$ and $v_i^* \otimes v_i^* \otimes \langle \xi|$ are eigenvectors of $\mathcal{M}(z)$. However, there are also the third eigenvectors in both proposition 3.1 and 3.2 which are slightly more involved. By using them we can generate further bilinear relations among the layer to

layer transfer matrices $S(z)_j^{\mathbf{a}}$'s mixing different boundary conditions \mathbf{a}, \mathbf{j} . To describe such relations we prepare some notation.

Recall that m and n are any positive integers representing the size of the layer as in (4.1). For a subset $I \subseteq \{1, \dots, m\}$ with the complement $\bar{I} = \{1, \dots, m\} \setminus I$ and sequences $\alpha \in \{0, 1\}^{\#I}$, $\beta \in \{0, 1\}^{\#\bar{I}}$, let $(\alpha_I, \beta_{\bar{I}}) \in \{0, 1\}^m$ be the sequence in which the subsequence corresponding to the indices in I is α and the rest \bar{I} is β . For instance for $m = 5$, $(\alpha_{\{1,3,4\}}, \beta_{\{2,5\}}) = (\alpha_1, \beta_1, \alpha_2, \alpha_3, \beta_2)$ for $I = \{1, 3, 4\}$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2)$ ⁶. Likewise for $J \sqcup \bar{J} = \{1, \dots, n\}$ and $\gamma \in \{0, 1\}^{\#J}$, $\delta \in \{0, 1\}^{\#\bar{J}}$, the symbol $(\gamma_J, \delta_{\bar{J}}) \in \{0, 1\}^n$ denotes the similar sequence. For any sequence $\alpha = (\alpha_1, \dots, \alpha_k) \in \{0, 1\}^k$, we set $|\alpha| = \alpha_1 + \dots + \alpha_k$ and $\bar{\alpha} = (1 - \alpha_1, \dots, 1 - \alpha_k)$.

Theorem 5.1 (Bilinear relations of layer to layer transfer matrices). *For any subsets $I \subseteq \{1, \dots, m\}$ and $J \subseteq \{1, \dots, n\}$ and sequences $\alpha \in \{0, 1\}^{\#I}$ and $\gamma \in \{0, 1\}^{\#J}$, we have (parentheses omitted in suffixes of S)*

$$\sum_{\beta, \delta} y^{|\beta|+|\delta|} x^{|\bar{\beta}|+|\bar{\delta}|} S(y)_{\gamma_J, \delta_{\bar{J}}}^{\alpha_I, \beta_{\bar{I}}} S(x)_{\gamma_J, \delta_{\bar{J}}}^{\alpha_I, \bar{\beta}_{\bar{I}}} = (x \longleftrightarrow y), \quad (5.1)$$

where the sum runs over $\beta \in \{0, 1\}^{\#\bar{I}}$ and $\delta \in \{0, 1\}^{\#\bar{J}}$.

Proposition 4.5 is the simplest case of theorem 5.1 corresponding to $I = \{1, \dots, m\}, J = \{1, \dots, n\}$, where the sum reduces to a single term. As another example, when $(m, n) = (4, 3), I = \{1, 3\}, J = \{2, 3\}, \alpha = (0, 1), \gamma = (1, 0)$, the relation (5.1) reads

$$\begin{aligned} & x^3 S(y)_{010}^{0010} S(x)_{110}^{0111} + yx^2 S(y)_{010}^{0011} S(x)_{110}^{0110} + yx^2 S(y)_{010}^{0110} S(x)_{110}^{0011} \\ & + y^2 x S(y)_{010}^{0111} S(x)_{110}^{0010} + yx^2 S(y)_{110}^{0010} S(x)_{010}^{0111} + y^2 x S(y)_{110}^{0011} S(x)_{010}^{0110} \\ & + y^2 x S(y)_{110}^{0110} S(x)_{010}^{0011} + y^3 S(y)_{110}^{0111} S(x)_{010}^{0010} = (x \longleftrightarrow y). \end{aligned}$$

We will present a proof of theorem 5.1 only for the special case considered in corollary 5.2 below, since the general case is easily inferred from it. It corresponds to the choice $I = \{2, 3, \dots, m\}, \alpha = \mathbf{a}, J = \{2, 3, \dots, n\}, \gamma = \mathbf{j}$ in (5.1), which suffices for our application to TASEP in the next section.

Corollary 5.2. *For any sequences $\mathbf{a} \in \{0, 1\}^{m-1}$ and $\mathbf{j} \in \{0, 1\}^{n-1}$, we have*

$$\begin{aligned} & x^2 S(y)_{0\mathbf{j}}^0 \mathbf{a} S(x)_{1\mathbf{j}}^1 \mathbf{a} + yx S(y)_{1\mathbf{j}}^0 \mathbf{a} S(x)_{0\mathbf{j}}^1 \mathbf{a} \\ & + yx S(y)_{0\mathbf{j}}^1 \mathbf{a} S(x)_{1\mathbf{j}}^0 \mathbf{a} + y^2 S(y)_{1\mathbf{j}}^1 \mathbf{a} S(x)_{0\mathbf{j}}^0 \mathbf{a} = (x \longleftrightarrow y). \end{aligned}$$

Proof. The proof proceeds similarly to that of proposition 4.5. Consider the following equality of operators on $F^{\otimes mn} \otimes F$.

$$\begin{aligned} & \sum_{\substack{\mathbf{b}, \mathbf{b}' \\ a_1'' + a_1''' = 1}} \mathcal{M}(\frac{x}{x'})_{a_1'', a_1'''}^{a_1, a_1'} \left(\mathcal{M}(\frac{y}{y'})_{b_n, b_n'}^{c_n, c_n'} \dots \mathcal{M}(\frac{y}{y'})_{b_1, b_1'}^{c_1, c_1'} \right) T(\frac{x}{y})_{i, \mathbf{j}}^{\mathbf{a}'', \mathbf{b}} T(\frac{x'}{y'})_{i', \mathbf{j}'}^{\mathbf{a}''', \mathbf{b}'} \\ & = \sum_{\substack{\mathbf{k}, \mathbf{k}' \\ j_1'' + j_1''' = 1}} T(\frac{x'}{y'})_{\mathbf{k}', \mathbf{j}'}^{\mathbf{a}', \mathbf{c}'} T(\frac{x}{y})_{\mathbf{k}, \mathbf{j}''}^{\mathbf{a}, \mathbf{c}} \mathcal{M}(\frac{y}{y'})_{j_1'', j_1'''}^{j_1', j_1'''} \left(\mathcal{M}(\frac{x}{x'})_{i_m, i_m'}^{k_m, k_m'} \dots \mathcal{M}(\frac{x}{x'})_{i_1, i_1'}^{k_1, k_1'} \right), \end{aligned} \quad (5.2)$$

where $\mathbf{a}, \mathbf{a}', \mathbf{a}'', \mathbf{a}'''$ (resp. $\mathbf{j}, \mathbf{j}', \mathbf{j}'', \mathbf{j}'''$)⁷ differ from each other only at the first component, which are a_1, a_1', a_1'', a_1''' (resp. j_1, j_1', j_1'', j_1'''). We take $a_1 + a_1' = 1$ and $j_1 + j_1' = 1$ and exhibited the constraints coming from (3.7). Unlike the previous (4.3) = (4.4), the identity operators were not written. The difference of (5.2) from (4.5) is that the pair (a_1, a_1) on the right end and (j_1, j_1) at the bottom left were changed to (a_1, a_1') and (j_1, j_1') which should necessarily be $(0, 1)$ or $(1, 0)$.

⁶ Note that it is *not* $(\alpha_1, \alpha_3, \alpha_4, \beta_2, \beta_5)$.

⁷ \mathbf{a}, \mathbf{j} here have a different meaning from those in the statement.

Substitution of $\alpha = xy'$, $\beta = x'y$ into (3.9) and (3.10) lead to

$$\sum_{i+j=1} \alpha^i \beta^j \mathcal{M}(\frac{y}{y'})_{i,j}^{k,l} |\chi(\frac{x}{x'})\rangle = \alpha^k \beta^l |\chi(\frac{x}{x'})\rangle, \quad \sum_{i+j=1} \alpha^i \beta^j \langle \chi(\frac{y'}{y}) | \mathcal{M}(\frac{x}{x'})_{k,l}^{i,j} = \alpha^k \beta^l \langle \chi(\frac{y'}{y}) |. \quad (5.3)$$

On the both sides of (5.2), multiply $\alpha^{a_1+j_1} \beta^{a'_1+j'_1}$ and take sum over $\mathbf{i}, \mathbf{i}', \mathbf{c}, \mathbf{c}'$ and a_1, a'_1, j_1, j'_1 with the constraints $a_1 + a'_1 = 1, j_1 + j'_1 = 1$. Evaluate the matrix element of the resulting operator identity between $\langle \chi(\frac{y'}{y}) |$ from the left and $|\chi(\frac{x}{x'})\rangle$ from the right. Thanks to the identities (3.11) and (5.3), all the M operators disappear. After canceling $\langle \chi(\frac{y'}{y}) | \chi(\frac{x}{x'}) \rangle \neq 0$ from the both sides we find

$$\begin{aligned} & \sum_{\substack{\mathbf{i}, \mathbf{i}', \mathbf{b}, \mathbf{b}' \\ a_1''+a_1'''=1, j_1+j_1'=1}} \alpha^{a_1''+j_1} \beta^{a_1''' + j_1'} T(\frac{x}{y})_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}'', \mathbf{b}} T(\frac{x'}{y'})_{\mathbf{i}', \mathbf{j}'}^{\mathbf{a}''', \mathbf{b}'} \\ &= \sum_{\substack{\mathbf{k}, \mathbf{k}', \mathbf{c}, \mathbf{c}' \\ a_1+a_1'=1, j_1'+j_1'''=1}} \alpha^{a_1+j_1''} \beta^{a_1'+j_1'''} T(\frac{x'}{y'})_{\mathbf{k}', \mathbf{j}'''}^{\mathbf{a}', \mathbf{c}'} T(\frac{x}{y})_{\mathbf{k}, \mathbf{j}''}^{\mathbf{a}, \mathbf{c}}. \end{aligned}$$

Using (4.1) and dividing by $(yy')^2$ we arrive at

$$\begin{aligned} & \sum_{a_1''+a_1'''=1, j_1+j_1'=1} (\frac{x}{y})^{a_1''+j_1} (\frac{x'}{y'})^{a_1''' + j_1'} S(\frac{x}{y})_{\mathbf{j}}^{\mathbf{a}''} S(\frac{x'}{y'})_{\mathbf{j}'}^{\mathbf{a}'''} \\ &= \sum_{a_1+a_1'=1, j_1'+j_1'''=1} (\frac{x}{y})^{a_1+j_1''} (\frac{x'}{y'})^{a_1'+j_1'''} S(\frac{x'}{y'})_{\mathbf{j}'''}^{\mathbf{a}'} S(\frac{x}{y})_{\mathbf{j}''}^{\mathbf{a}}, \end{aligned}$$

as desired. \square

Example 5.3. When $(m, n) = (1, 2)$ and $\mathbf{j} = (0)$, corollary 5.2 says

$$x^2 S(y)_{00}^1 S(x)_{10}^1 + yx S(y)_{10}^0 S(x)_{00}^1 + yx S(y)_{00}^1 S(x)_{10}^0 + y^2 S(y)_{10}^1 S(x)_{00}^0 = (x \longleftrightarrow y).$$

In fact substituting example 4.2 and using (3.1) we find that the left hand side is equal to

$$q(x+y)(\mathbf{a}^- \otimes \mathbf{k}) + q(x^2 + y^2)(1 \otimes \mathbf{k}) + qxy(x+y)(q(1-q)\mathbf{k} \otimes \mathbf{a}^+ \mathbf{k} + \mathbf{a}^+ \otimes \mathbf{k}) + xyW$$

for some $W \in \mathcal{A}^{\otimes 2}$ independent of x and y .

Remark 5.4. One can generalize the bilinear relation in theorem 5.1 further by introducing *inhomogeneity parameters* as follows. In (4.1) we consider horizontal lines as carrying parameters x_1, \dots, x_m from the top to the bottom and vertical lines y_1, \dots, y_n from the left to the right. Set $\mathbf{x} = (x_1, \dots, x_m), \mathbf{y} = (y_1, \dots, y_n)$. We define $S(\mathbf{x}; \mathbf{y})_{\mathbf{j}}^{\mathbf{a}}$ by changing the parameter z of $\mathcal{L}(z)$ to x_i/y_j if this $\mathcal{L}(z)$ is situated on the vertex where the i -th horizontal and the j -th vertical line meet. As in theorem 5.1 let I, J be subsets of $\{1, \dots, m\}, \{1, \dots, n\}$ and take $\alpha \in \{0, 1\}^{\#I}, \gamma \in \{0, 1\}^{\#J}$. Suppose that $(\mathbf{x}; \mathbf{y})$ and $(\mathbf{x}'; \mathbf{y}')$ satisfy

$$x_1/x'_1 = \dots = x_m/x'_m = u, \quad y_1/y'_1 = \dots = y_n/y'_n = v. \quad (5.4)$$

In other words, x_i/x'_i or y_i/y'_i do not depend on i . Then as a generalization of theorem 5.1 we have

$$\sum_{\beta, \delta} (\frac{u}{v})^{|\beta|+|\delta|} S(\mathbf{x}; \mathbf{y})_{\gamma, \delta}^{\alpha, \beta} S(\mathbf{x}'; \mathbf{y}')_{\gamma, \delta}^{\alpha, \bar{\beta}} = \sum_{\beta, \delta} (\frac{u}{v})^{|\beta|+|\delta|} S(\mathbf{x}'; \mathbf{y}')_{\gamma, \delta}^{\alpha, \beta} S(\mathbf{x}; \mathbf{y})_{\gamma, \delta}^{\alpha, \bar{\beta}}, \quad (5.5)$$

where the sums are over $\beta \in \{0, 1\}^{\#I}$ and $\delta \in \{0, 1\}^{\#J}$ as in (5.1).

$$\begin{aligned}
& \sum_{\substack{\mathbf{b}, \mathbf{b}' \\ a_1'' + a_1''' = 1}} \mathcal{M}(\frac{x_1}{x_1'})^{a_{a_1'', a_1'''} } \left(\mathcal{M}(\frac{y_n}{y_n'})^{c_n, c_n'} \dots \mathcal{M}(\frac{y_1}{y_1'})^{c_1, c_1'} \right) T(\mathbf{x}; \mathbf{y})_{\mathbf{i}, \mathbf{j}}^{\mathbf{a}'', \mathbf{b}} T(\mathbf{x}'; \mathbf{y}')_{\mathbf{i}', \mathbf{j}'}^{\mathbf{a}''', \mathbf{b}'} \\
&= \sum_{\substack{\mathbf{k}, \mathbf{k}' \\ j_1'' + j_1''' = 1}} T(\mathbf{x}'; \mathbf{y}')_{\mathbf{k}', \mathbf{j}'''}^{\mathbf{a}', \mathbf{c}'} T(\mathbf{x}; \mathbf{y})_{\mathbf{k}, \mathbf{j}''}^{\mathbf{a}, \mathbf{c}} \mathcal{M}(\frac{y_1}{y_1'})_{j_1'', j_1'''}^{j_1'', j_1'''} \left(\mathcal{M}(\frac{x_m}{x_m'})_{i_m, i_m'}^{k_m, k_m'} \dots \mathcal{M}(\frac{x_1}{x_1'})_{i_1, i_1'}^{k_1, k_1'} \right).
\end{aligned}$$

6. APPLICATION TO n -TASEP: PROOF OF THEOREM 2.2

The figure shows five 2D coordinate systems, each with a horizontal x-axis and a vertical y-axis. In each system, a red vector is drawn from the origin (0,0) to the point (1,1). The black axes are labeled with 0 at the origin and 1 at the unit distance along each axis. The red vector represents the sum of the two unit vectors.

Define the operators $X_0(z), \dots, X_n(z)$ by

$$X_0(z) = \begin{array}{c} \uparrow \uparrow \uparrow \\ | \\ \downarrow \end{array} + \begin{array}{c} \color{red}\uparrow \uparrow \uparrow \\ | \\ \downarrow \end{array} + \begin{array}{c} \color{red}\uparrow \uparrow \uparrow \\ | \\ \color{red}\downarrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ \color{red}| \\ \downarrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ \color{red}| \\ \color{red}\downarrow \end{array}$$

$$= 1 \otimes 1 \otimes 1 + z \mathbf{a}^+ \otimes 1 \otimes 1 + z \mathbf{k} \otimes \mathbf{a}^+ \otimes 1 + z \mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{a}^+ + z^2 1 \otimes \mathbf{a}^+ \otimes \mathbf{a}^+,$$

$$\begin{aligned}
X_1(z) &= \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} \\
&= z\mathbf{k} \otimes \mathbf{k} \otimes 1 + z\mathbf{a}^- \otimes \mathbf{k} \otimes \mathbf{a}^+ + z^2 1 \otimes \mathbf{k} \otimes \mathbf{a}^+, \\
X_2(z) &= \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} \\
&= z1 \otimes \mathbf{a}^- \otimes \mathbf{k} + z^2 \mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{k} + z^2 \mathbf{k} \otimes 1 \otimes \mathbf{k}, \\
X_3(z) &= \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} \\
&= z1 \otimes \mathbf{a}^- \otimes \mathbf{a}^- + z^2 \mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{a}^- + z^2 \mathbf{k} \otimes 1 \otimes \mathbf{a}^- + z^2 \mathbf{a}^- \otimes 1 \otimes 1 + z^3 1 \otimes 1 \otimes 1.
\end{aligned}$$

Proposition 6.3. *The operators $X_i(z)$'s are contained in the layer to layer transfer matrices at $q = 0$ as follows:*

$$S(z)_{00 \dots 0}^{00 \dots 0} = \sum_{i=0}^n X_i(z) \otimes \underbrace{\mathbf{a}^+ \otimes \dots \otimes \mathbf{a}^+}_i \otimes \underbrace{1 \otimes \dots \otimes 1}_{n-i} \otimes 1 \otimes \dots \otimes 1, \quad (6.3)$$

diagonal

$$S(z)_{10 \dots 0}^{10 \dots 0} = z^{-1} \sum_{i=0}^n X_i(z) \otimes \underbrace{1 \otimes \dots \otimes 1}_i \otimes \underbrace{\mathbf{a}^- \otimes \dots \otimes \mathbf{a}^-}_{n-i} \otimes \underbrace{\mathbf{a}^+ \otimes \dots \otimes \mathbf{a}^+}_{n-1} \otimes 1 \otimes \dots \otimes 1. \quad (6.4)$$

diagonal

Here ‘diagonal’ signifies the part of the tensor components corresponding to the vertices on the NE-SW diagonal in (4.1) with $m = n$.

Proof. We regard the triangle shape region in (6.2) as embedded into the $n \times n$ square lattice in (4.1) $_{m=n}$. When $q = 0$, the rightmost vertex of $\mathcal{L}(z)$ in (3.6) is absent. This means that the red lines for the allowed configurations tend to be confined in the upper left region. Also, once an edge on the SW-NE boundary in (6.2) becomes black, then the subsequent ones continue to be black in its further NE. These properties imply the claimed expansion formulas. See the following example from $n = 3$, where black and red edges are fixed to 0 and 1 respectively, whereas the dotted ones are to be summed over 0 and 1⁸. The four diagrams correspond to $i = 0, \dots, 3$ terms in (6.3) and (6.4) from the left to the right. General case is similar.

$$\begin{aligned}
S(z)_{000}^{000} &= \sum \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \sum \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \sum \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \sum \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} \\
S(z)_{100}^{100} &= \sum \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \sum \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \sum \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array} + \sum \begin{array}{c} \uparrow \uparrow \uparrow \\ | \quad | \quad | \\ \text{---} \quad \text{---} \quad \text{---} \\ | \quad | \quad | \\ \uparrow \uparrow \uparrow \end{array}
\end{aligned}$$

For the weight of z , notice that it is calculated by $\#(1 \text{ on the top edges}) - \#(1 \text{ on the bottom edges})$. \square

⁸ Some of them are actually fixed to 0 or 1 by (6.1), but they are left dotted for the sake of exposition.

Example 6.4. Consider the case $n = 2$. Setting $q = 0$ in example 4.3, we have

$$\begin{aligned} S(z)_{00}^{00} &= (1 + z\mathbf{a}^+) \otimes 1 \otimes 1 \otimes 1 + z\mathbf{k} \otimes \mathbf{a}^+ \otimes 1 \otimes 1 + (z\mathbf{a}^- + z^2 1) \otimes \mathbf{a}^+ \otimes \mathbf{a}^+ \otimes 1 \\ &= X_0(z) \otimes 1 \otimes 1 \otimes 1 + X_1(z) \otimes \mathbf{a}^+ \otimes 1 \otimes 1 + X_2(z) \otimes \mathbf{a}^+ \otimes \mathbf{a}^+ \otimes 1 \end{aligned}$$

by example 6.1 in agreement with (6.3). Similarly example 4.4 leads to

$$\begin{aligned} zS(z)_{10}^{10} &= (1 + z\mathbf{a}^+) \otimes \mathbf{a}^- \otimes \mathbf{a}^- \otimes \mathbf{a}^+ + z\mathbf{k} \otimes 1 \otimes \mathbf{a}^- \otimes \mathbf{a}^+ + (z\mathbf{a}^- + z^2 1) \otimes 1 \otimes 1 \otimes \mathbf{a}^+ \\ &= X_0(z) \otimes \mathbf{a}^- \otimes \mathbf{a}^- \otimes \mathbf{a}^+ + X_1(z) \otimes 1 \otimes \mathbf{a}^- \otimes \mathbf{a}^+ + X_2(z) \otimes 1 \otimes 1 \otimes \mathbf{a}^+ \end{aligned}$$

in agreement with (6.4).

Now we are going to extract the relations among $X_i(z)$'s from the $q = 0$ limit of the bilinear identities in proposition 4.5 and corollary 5.2.

Proposition 6.5 (Difference analogue of the hat relation). *The operators $X_i(z)$'s satisfy the following relations:*

$$[X_i(x), X_j(y)] = [X_i(y), X_j(x)] \quad (0 \leq i, j \leq n), \quad (6.5)$$

$$xX_i(y)X_j(x) = yX_i(x)X_j(y) \quad (0 \leq j < i \leq n). \quad (6.6)$$

Proof. Substituting (6.3) into (4.2) and taking the coefficient of

$$\overbrace{(\mathbf{a}^+)^2 \otimes \cdots \otimes (\mathbf{a}^+)^2}^j \otimes \overbrace{\mathbf{a}^+ \otimes \cdots \otimes \mathbf{a}^+}^{i-j} \otimes 1 \otimes \cdots \otimes 1 \quad (0 \leq j \leq i \leq n),$$

we get (6.5). Set $\mathbf{a} = (0, \dots, 0)$, $\mathbf{j} = (0, \dots, 0)$ in corollary 5.2 and use the obvious property $S(z)_{00 \dots 0}^{10 \dots 0} = S(z)_{10 \dots 0}^{00 \dots 0} = 0$ to derive

$$x^2 S(y)_{00 \dots 0}^{00 \dots 0} S(x)_{10 \dots 0}^{10 \dots 0} + y^2 S(y)_{10 \dots 0}^{10 \dots 0} S(x)_{00 \dots 0}^{00 \dots 0} = (x \longleftrightarrow y).$$

Substitute (6.3), (6.4) into this and take the coefficient of

$$\overbrace{\mathbf{a}^+ \otimes \cdots \otimes \mathbf{a}^+}^j \otimes \overbrace{\mathbf{k} \otimes \cdots \otimes \mathbf{k}}^{i-j} \otimes \overbrace{\mathbf{a}^- \otimes \cdots \otimes \mathbf{a}^-}^{n-i} \otimes (\text{off diagonal}) \quad (0 \leq j < i \leq n).$$

Noting that such term comes only from $(\overbrace{\mathbf{a}^+ \otimes \cdots \otimes \mathbf{a}^+}^i \otimes 1 \otimes \cdots \otimes 1)(\overbrace{1 \otimes \cdots \otimes 1}^j \otimes \mathbf{a}^- \otimes \cdots \otimes \mathbf{a}^-)$, we obtain (6.6). \square

Remark 6.6. The relations in proposition 6.5 are rearranged as

$$X_i(x)X_j(y) = \begin{cases} X_i(y)X_j(x) + (1 - \frac{x}{y})X_j(y)X_i(x) & i < j, \\ X_i(y)X_i(x) & i = j, \\ \frac{x}{y}X_i(y)X_j(x) & i > j. \end{cases}$$

This exchange rule satisfies the Yang-Baxter relation in that the two ways of rewriting $X_i(x)X_j(y)X_k(z)$ as linear combinations of $X_{k'}(z)X_{j'}(y)X_{i'}(x)$ with $\{i', j', k'\} = \{i, j, k\}$ lead to the identical result. They are equivalent to the $t = 0$ case of eqs. (25) and (26) in [6] under the formal correspondence $X_i(z) = A_{n-i}(z^{-1})$.

Finally we introduce the n -TASEP operators $X_i, \hat{X}_i \in (\mathcal{A}_{q=0})^{\otimes n(n-1)/2}$ [10] by

$$X_i = X_i(z = 1), \quad \hat{X}_i = \frac{d}{dz} X_i(z)|_{z=1} \quad (0 \leq i \leq n). \quad (6.7)$$

From (6.2) we see that they coincide with those defined in (2.7) as the configuration sums of the 0-oscillator valued five-vertex model whose vertices are specified in (2.8).

Proof of theorem 2.2. Differentiate (6.5) and (6.6) with respect to y and set $x, y = 1$. \square

7. SUMMARY

In this paper we have proved the hat relation in theorem 2.2 among the operators X_i and \hat{X}_i defined by (2.7). It provides an alternative derivation of the matrix product formula for the steady state probability (2.4) of the n -TASEP, which was obtained earlier in [10] by identifying the Ferrari-Martin algorithm with a composition of the combinatorial R .

Reversing the order of presentation in this paper, our proof of the hat relation may be summarized as follows. The hat relation (theorem 2.2) is first upgraded to the difference analogue in proposition 6.5. By introducing q and embedding into the 3D lattice model, it is further upgraded to bilinear relations among layer to layer transfer matrices (theorem 5.1). Finally these relations are attributed to the most local property, the tetrahedron equation in proposition 3.4.

The present paper and [10] reveal a hidden 3D integrable structure in the multispecies TASEP. It deserves further investigation whether such results can be generalized to the large list of matrix product constructions of the quantum and combinatorial R by the tetrahedron equation [11, 9]. It turns out that another prototype model of stochastic dynamics known as the multispecies *totally asymmetric zero range process* (so called TAZRP) can be analyzed in a completely similar manner based on the scheme given in this paper. We plan to present the detail in a future publication.

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